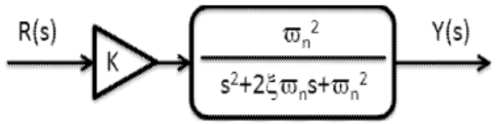
6. Second Order Design

EN 2142 Electronic Control Systems

Dr. Rohan Munasinghe
BSc, MSc, PhD, MIEEE
Department of Electronic and Telecommunication Engineering
Faculty of Engineering
University of Moratuwa 10400

Second Order Systems

- General second order system



ω_n : natural undamped frequency
ς : damping coefficient

- Transfer function $G_{g2}(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

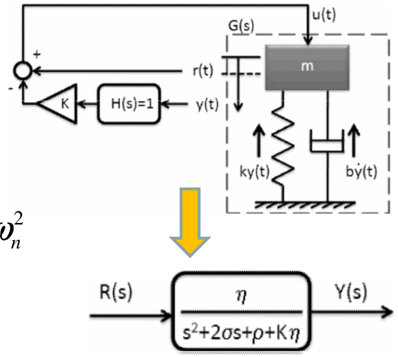
- Eg: Height adjustable stabilized platform

$$G_c(s) = \frac{\eta/(s^2+2\sigma s+\rho)}{1+K\eta/(s^2+2\sigma s+\rho)} = \frac{\eta}{s^2+2\sigma s+\rho+K\eta}$$

where $\eta = 1/m, \sigma = \frac{b}{2m}$ and $\rho = k/m$

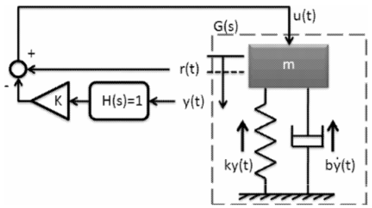
$$\frac{Y(s)}{R(s)} = \frac{1}{k+K}\cdot\frac{(k+K)/m}{s^2+(b/m)s+(k+K)/m}$$

$(k+K)/m \rightarrow \omega_n^2$, $(b/m) \rightarrow 2\varsigma\omega$



ζ and ω_n

m, k, b → $\omega_n = \sqrt{\dfrac{k+K}{m}}$, $\zeta = \dfrac{b}{2\sqrt{m(k+K)}}$



- DCG of the Generic 2nd order system

$$\frac{Y(s)}{R(s)} = \frac{1}{k+K}\,\frac{(k+K)/m}{s^2+(b/m)s+(k+K)/m}$$

$$DCG = \lim_{s\to 0} s\,\frac{(k+K)/m}{s^2+(b/m)s+(k+K)/m} = 1$$

$$DCG = \frac{1}{k+K}$$

ζ Determines the Response

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

- Poles $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$

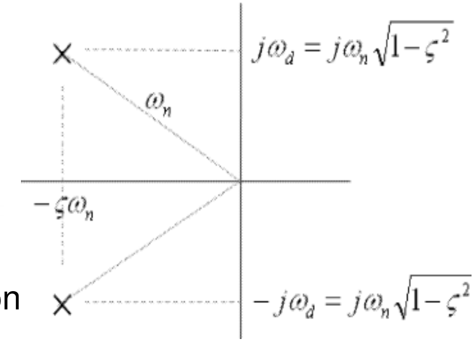
- Response

1. ζ >1 over damped (slow, no oscillations)
2. ζ =1 critically damped (quickest, nonoscillatory)
3. 0< ζ <1 under damped (damped oscillations)
4. ζ =0 stable sustained oscillations (simple harmonic motion)
5. ζ <0 unstable response

0<ζ<1 Most Common In Industry

- Under Damped Response
- Poles

$$\begin{aligned} s_1, s_2 &= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \\ &= -\zeta\omega_n \pm j\omega_d \end{aligned}$$



- Under damped transfer function

$$\begin{aligned} G_2(s) &= \frac{\omega_n^2}{(s+\zeta\omega_n+j\omega_d)(s+\zeta\omega_n-j\omega_d)} \\ &= \frac{\omega_n^2}{(s+\zeta\omega_n)^2+\omega_d^2} \\ &= \frac{\omega_n^2}{\omega_d}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2} \end{aligned}$$

Unit Step Response (0<ζ<1)

$$Y(s) = \frac{\omega_n^2}{\omega_d}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\cdot\frac{1}{s}$$

$$y(t) = \int e^{-\zeta\omega_n t}\sin(\omega_d t)dt$$

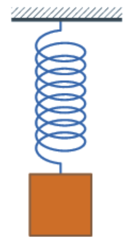
Exponential scaling in time

Integration

$$\phi = \frac{\omega_d}{\zeta\omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

Appendix B

$$y(t) = \frac{\omega_n^2}{\omega_d}\left[\frac{\omega_d}{\omega_d^2+\zeta^2\omega_n^2} - \frac{e^{-\zeta\omega_n t}}{\sqrt{\omega_d^2+\zeta^2\omega_n^2}}\sin(\omega_d t+\phi)\right]$$



- Because $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\begin{aligned} y(t) &= \frac{\omega_n^2}{\omega_d}\left[\frac{\omega_d}{\omega_n^2} - \frac{e^{-\zeta\omega_n t}}{\omega_n}\sin(\omega_d t+\phi)\right] \\ &= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\phi) \end{aligned}$$

Peak time

Peak time t_p is the time the response takes to reach its first peak

$$\begin{aligned} \dot{y}(t) &= -\frac{1}{\sqrt{1-\zeta^2}}\left[\omega_d e^{-\zeta\omega_n t}\cos(\omega_d t+\phi) - \sin(\omega_d t+\phi)e^{-\zeta\omega_n t}\zeta\omega_n\right] \\ &= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\zeta\sin(\omega_d t+\phi) - \sqrt{1-\zeta^2}\cos(\omega_d t+\phi)\right] \\ &= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t+\phi-\phi) \\ &= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t) \qquad (5.11) \end{aligned}$$

when $t = t_p$ $\dot{y}(t)=0$, therefore $\sin(\omega_d t_p)=0$

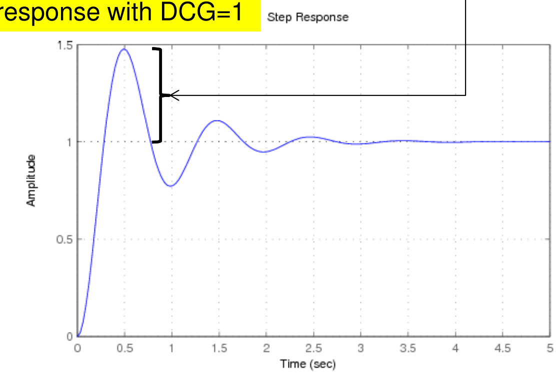
$$\begin{aligned} \omega_d t_p &= \pi \\ \omega_n\sqrt{1-\zeta^2}t_p &= \pi \\ t_p &= \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \end{aligned}$$

Rise Time and Overshoot

- Time duration between 10%-90% of the first peak of the damped oscillation
- Rule of thumb assumong ζ=0.5 ⟶ $t_r \approx \frac{1.8}{\omega_n}$

$$\text{at } t = t_p \quad 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p}\sin(\omega_d t_p+\phi) = 1 + PO$$

Generic 2nd order response with DCG=1



Overshoot

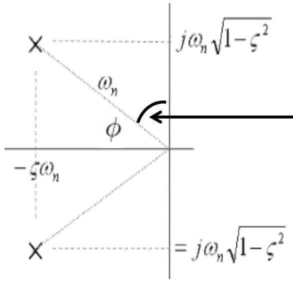
$$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p}\sin\left(\omega_d t_p+\phi\right)=1+PO$$

$\omega_n t_p = \frac{\pi}{\sqrt{1-\zeta^2}}$ $\quad$ $\omega_d t_p = \pi$

$$-\frac{1}{\sqrt{1-\zeta^2}}e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}\sin\left(\pi+\phi\right)=PO$$

$\sin\pi+\phi = -\sin\phi = \sqrt{1-\zeta^2}$

$$\begin{aligned}
\frac{1}{\sqrt{1-\zeta^2}}e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}\sqrt{1-\zeta^2} &= PO \\
e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} &= PO \\
-\frac{\zeta\pi}{\sqrt{1-\zeta^2}} &= \ln PO \\
\tan\beta &= -\frac{\ln PO}{\pi} \\
\beta &= \tan^{-1}\left\{-\frac{\ln PO}{\pi}\right\}
\end{aligned}$$



$\beta = \pi/2 - \phi$

Settling Time

- Time for amplitude t drop to 1% of the original amplitude

$$y(t) = 1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\phi)$$

$$\begin{aligned}
e^{-\zeta\omega_n t_s} &\approx 0.01 \\
-\zeta\omega_n t_s &\approx \ln 0.01 \\
t_s &\approx \frac{4.6}{\zeta\omega_n}
\end{aligned}$$

Design Conditions

maximum overshoot PO_{max} is specified

$$\beta > \tan^{-1}\left\{-\frac{\ln PO_{max}}{\pi}\right\}$$

maximum settling time $t_{s,max}$ is specified

$$\zeta\omega_n > \frac{4.6}{t_{s,max}}$$

response has to rise before a maximum time of $t_{r,max}$

$$\omega_n > \frac{1.8}{t_{r,max}}$$